Complete factorial designs

Session 5

MATH 80667A: Experimental Design and Statistical Methods
HEC Montréal

Outline

Factorial designs and interactions

Tests for two-way ANOVA

Factorial designs and interactions

Complete factorial designs?

Factorial design

study with multiple factors (subgroups)

Complete

Gather observations for every subgroup

Motivating example

Response:
retention of information
two hours after reading a story

Population:
children aged four

experimental factor 1:
ending (happy or sad)

experimental factor 2:
complexity (easy, average or hard).

Setup of design

complexity	happy	sad
complicated	😆	😞
average	😆	😞
easy	😆	😞

Efficiency of factorial design

Cast problem as a series of one-way ANOVA vs simultaneous estimation

Factorial designs requires **fewer overall observations**

Can study **interactions**

Interaction

Definition: when the effect of one factor depends on the levels of another factor.

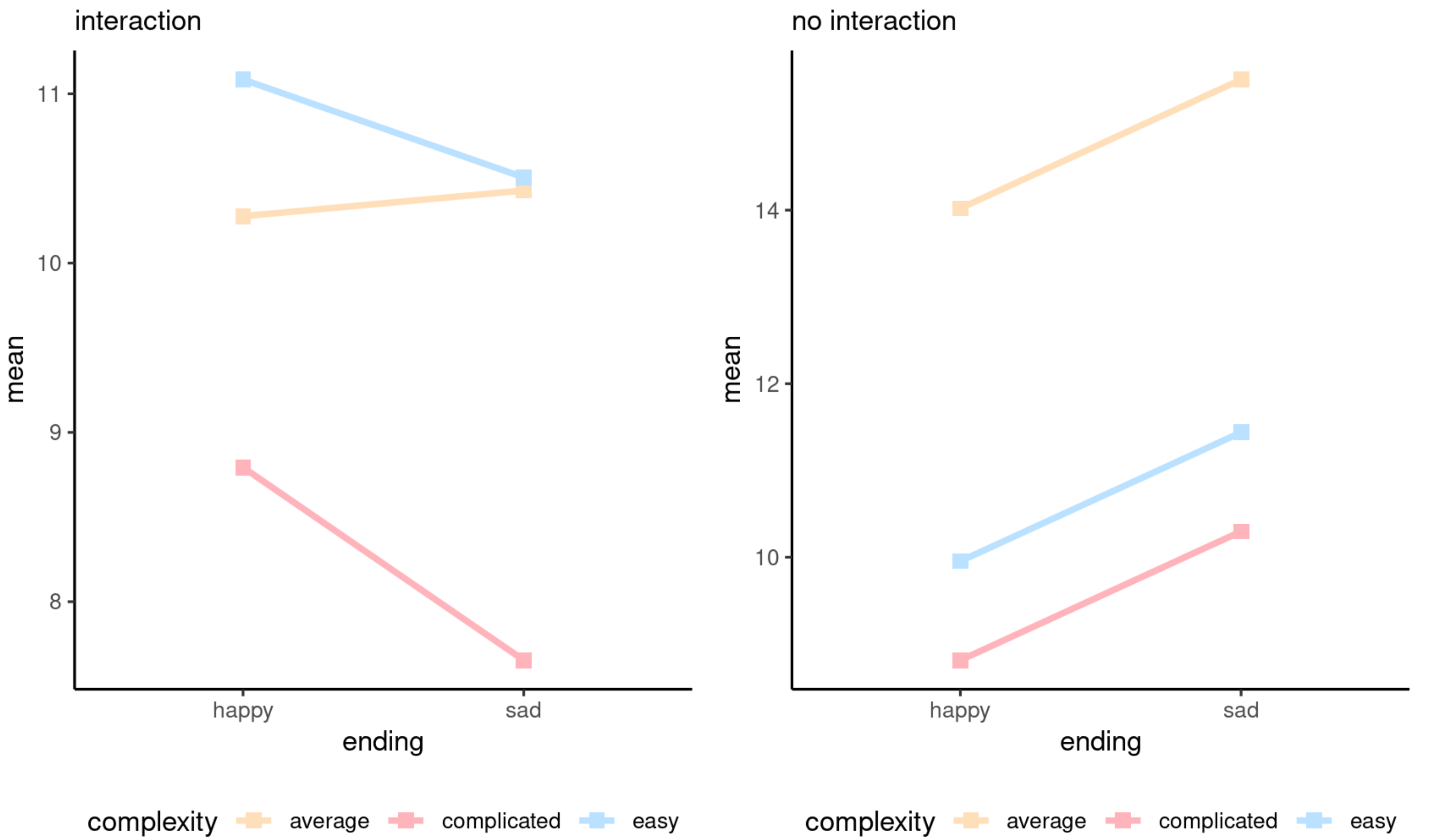
Effect together
$\neq$
sum of individual effects

Interaction or profile plot

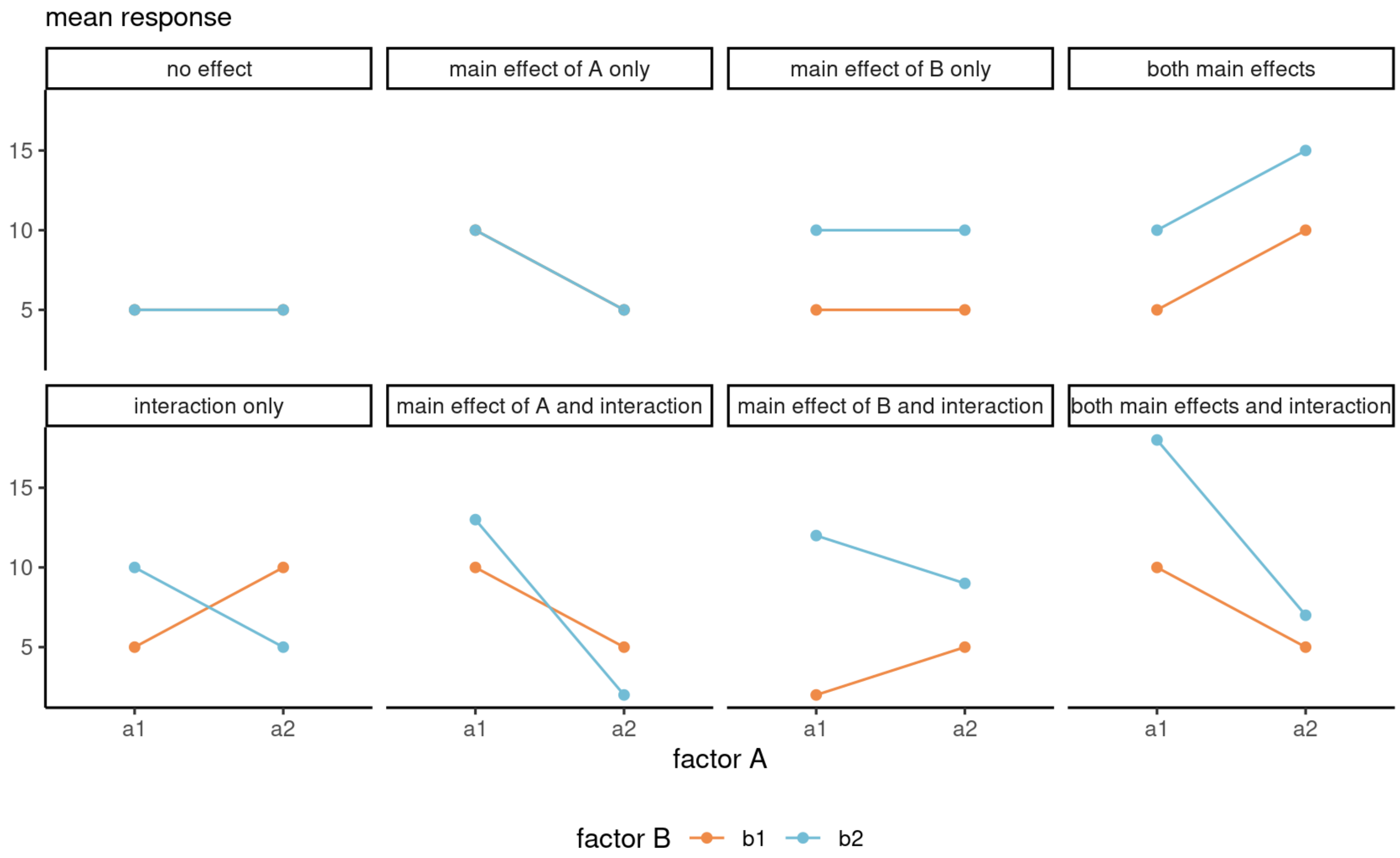
Graphical display:
plot sample mean per category

with uncertainty measure
(1 std. error for mean
confidence interval, etc.)

Interaction plots and parallel lines



Interaction plots for 2 by 2 designs



Cell means for 2 by 2 designs

	b1	b2
a1	5	5
a2	5	5

	b1	b2
a1	10	10
a2	5	5

	b1	b2
a1	5	10
a2	5	10

	b1	b2
a1	5	10
a2	10	15

	b1	b2
a1	5	10
a2	10	5

	b1	b2
a1	10	13
a2	5	2

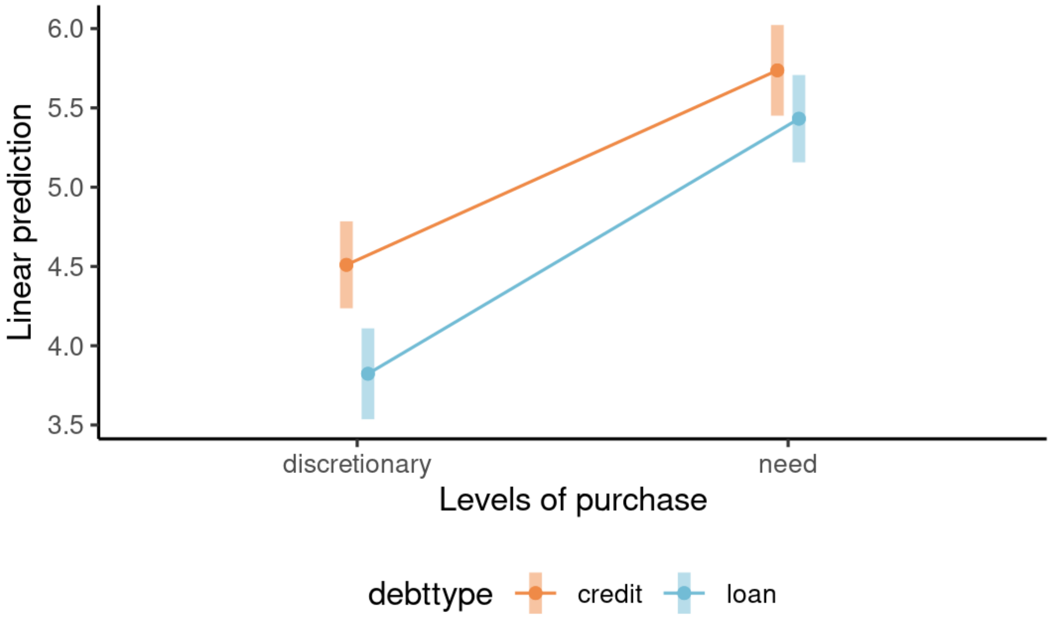
	b1	b2
a1	2	12
a2	5	9

	b1	b2
a1	10	18
a2	5	7

Example 1 : loans versus credit

Sharma, Tully, and Cryder (2021)
Supplementary study 5 consists of a 2×2 between-subject ANOVA with factors

- debt type (`debttype`), either "loan" or "credit"
- `purchase` type, either `discretionary` or not (`need`)

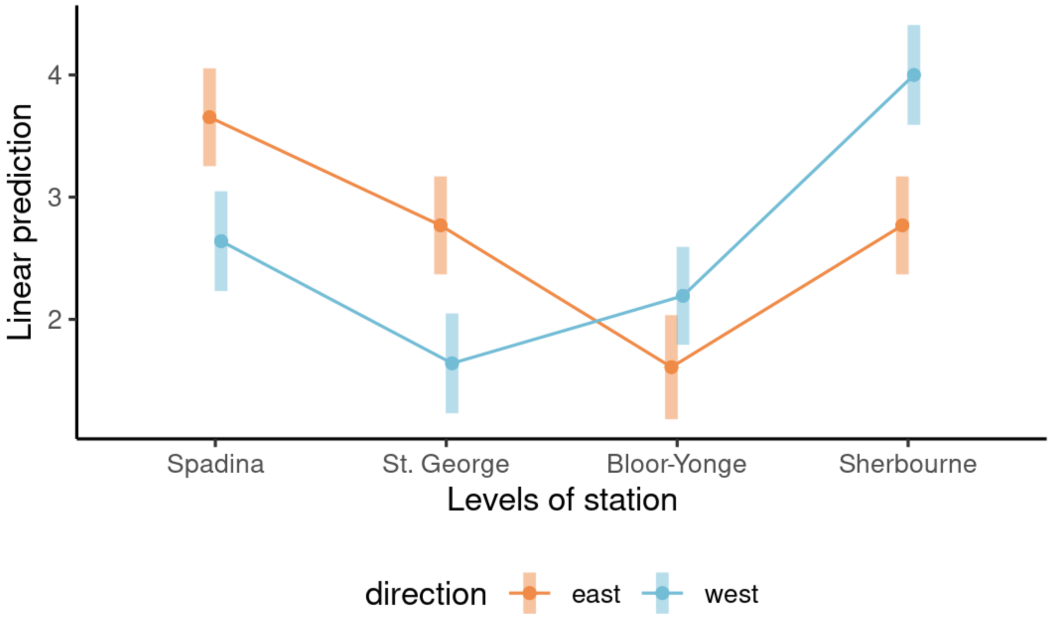


No evidence of interaction

Example 2 - psychological distance

Maglio and Polman (2014) Study 1 uses a 4×2 between-subject ANOVA with factors

- subway `station`, one of Spadina, St. George, Bloor-Yonge and Sherbourne
- `direction` of travel, either east or west



Clear evidence of interaction (symmetry?)

Tests for two-way ANOVA

Analysis of variance = regression

An analysis of variance model is simply a **linear regression** with categorical covariate(s).

- Typically, the parametrization is chosen so that parameters reflect differences to the global mean (sum-to-zero parametrization).
- The full model includes interactions between all combinations of factors
  - one average for each subcategory
  - one-way ANOVA!

Formulation of the two-way ANOVA

Two factors: A (complexity) and B (ending) with $n_a = 3$ and $n_b = 2$ levels, and their interaction.

Write the average response Y_{ijr} of the rth measurement in group (a_i, b_j) as

$$\underbrace{\mathsf{E}(Y_{ijr})}_{\text{average response}} = \underbrace{\mu_{ij}}_{\text{subgroup mean}}$$

where Y_{ijr} are independent observations with a common std. deviation σ.

- We estimate μ_{ij} by the sample mean of the subgroup (i, j), say $\widehat{\mu}_{ij}$.
- The fitted values are $\hat{y}_{ijr} = \widehat{\mu}_{ij}$.

One average for each subgroup

B **ending** A **complexity**	b_1 **(happy)**	b_2 **(sad)**	***row mean***
a_1 (complicated)	μ_{11}	μ_{12}	$\mu_{1.}$
a_2 (average)	μ_{21}	μ_{22}	$\mu_{2.}$
a_3 (easy)	μ_{31}	μ_{32}	$\mu_{3.}$
column mean	$\mu_{.1}$	$\mu_{.2}$	μ

Row, column and overall average

- Mean of A_i (average of row i):

$$\mu_{i.} = \frac{\mu_{i1} + \cdots + \mu_{in_b}}{n_b}$$

- Mean of B_j (average of column j):

$$\mu_{.j} = \frac{\mu_{1j} + \cdots + \mu_{n_a j}}{n_a}$$

- Overall average:

$$\mu = \frac{\sum_{i=1}^{n_a} \sum_{j=1}^{n_b} \mu_{ij}}{n_a n_b}$$

- Row, column and overall averages are **equiweighted** combinations of the cell means μ_{ij}.
- Estimates are obtained by replacing μ_{ij} in formulas by subgroup sample mean.

Vocabulary of effects

- **simple effects**: difference between levels of one in a fixed combination of others (change in difficulty for happy ending)
- **main effects**: differences relative to average for each condition of a factor (happy vs sad ending)
- **interaction effects**: when simple effects differ depending on levels of another factor

Main effects

Main effects are comparisons between row or column averages

Obtained by *marginalization*, i.e., averaging over the other dimension.

Main effects are not of interest if there is an interaction.

	happy	sad
column means	$\mu_{.1}$	$\mu_{.2}$

complexity	row means
complicated	$\mu_{1.}$
average	$\mu_{2.}$
easy	$\mu_{3.}$

Simple effects

Simple effects are comparisons between cell averages within a given row or column

	happy	sad
means (easy)	μ_{13}	μ_{23}

complexity	mean (happy)
complicated	μ_{11}
average	μ_{21}
easy	μ_{31}

Contrasts

We collapse categories to obtain a one-way ANOVA with categories A (complexity) and B (ending).

Q: How would you write the weights for contrasts for testing the

- main effect of A: complicated vs average, or complicated vs easy.
- main effect of B: happy vs sad.
- interaction A and B: difference between complicated and average, for happy versus sad?

The order of the categories is (a_1, b_1), (a_1, b_2), ..., (a_3, b_2).

Contrasts

Suppose the order of the coefficients is factor A (complexity, 3 levels, complicated/average/easy) and factor B (ending, 2 levels, happy/sad).

test	μ_{11}	μ_{12}	μ_{21}	μ_{22}	μ_{31}	μ_{32}
main effect A (complicated vs average)	1	1	-1	-1	0	0
main effect A (complicated vs easy)	1	1	0	0	-1	-1
main effect B (happy vs sad)	1	-1	1	-1	1	-1
interaction AB (comp. vs av, happy vs sad)	1	-1	-1	1	0	0
interaction AB (comp. vs easy, happy vs sad)	1	-1	0	0	-1	1

Global hypothesis tests

Main effect of factor A

$\mathscr{H}_0$: $\mu_{1.} = \cdots = \mu_{n_a.}$ vs $\mathscr{H}_a$: at least two marginal means of A are different

Main effect of factor B

$\mathscr{H}_0$: $\mu_{.1} = \cdots = \mu_{.n_b}$ vs $\mathscr{H}_a$: at least two marginal means of B are different

Interaction

$\mathscr{H}_0$: $\mu_{ij} = \mu_{i.} + \mu_{.j}$ (sum of main effects) vs $\mathscr{H}_a$: effect is not a combination of row/column effect.

Comparing nested models

Rather than present the specifics of ANOVA, we consider a general hypothesis testing framework which is more widely applicable.

We compare two competing models, $\mathbb{M}_a$ and $\mathbb{M}_0$.

- the **alternative** or full model $\mathbb{M}_a$ under the alternative $\mathscr{H}_a$ with p parameters for the mean
- the simpler **null** model $\mathbb{M}_0$ under the null $\mathscr{H}_0$, which imposes ν restrictions on the full model

Intuition behind F-test for ANOVA

The **residual sum of squares** measures how much variability is leftover,

$$\mathsf{RSS}_a = \sum_{i=1}^{n} \left(y_i - \hat{y}_i^{\mathbb{M}_a}\right)^2$$

where $\hat{y}_i$ is the estimated mean under model $\mathbb{M}_a$ for the observation y_i.

The more complex fits better (it is necessarily more flexible), but requires estimation of more parameters.

- We wish to assess the improvement that would occur by chance, if the null model was correct.

Testing linear restrictions in linear models

If the alternative model has p parameters for the mean, and we impose ν linear restrictions under the null hypothesis to the model estimated based on n independent observations, the test statistic is

$$F = \frac{(\mathsf{RSS}_0 - \mathsf{RSS}_a)/\nu}{\mathsf{RSS}_a/(n-p)}$$

- The numerator is the difference in residuals sum of squares, denoted RSS, from models fitted under $\mathscr{H}_0$ and $\mathscr{H}_a$, divided by degrees of freedom ν.
- The denominator is an estimator of the variance, obtained under $\mathscr{H}_a$ (termed mean squared error of residuals)
- The benchmark for tests in linear models is Fisher's $\mathsf{F}(\nu, n-p)$.

Analysis of variance table

term	degrees of freedom	mean square	F
A	$n_a - 1$	$\mathsf{MS}_A = \mathsf{SS}_A/(n_a - 1)$	$\mathsf{MS}_A/\mathsf{MS}_{\text{res}}$
B	$n_b - 1$	$\mathsf{MS}_B = \mathsf{SS}_B/(n_b - 1)$	$\mathsf{MS}_B/\mathsf{MS}_{\text{res}}$
AB	$(n_a - 1)(n_b - 1)$	$\mathsf{MS}_{AB} = \mathsf{SS}_{AB}/\{(n_a - 1)(n_b - 1)\}$	$\mathsf{MS}_{AB}/\mathsf{MS}_{\text{res}}$
residuals	$n - n_a n_b$	$\mathsf{MS}_{\text{resid}} = \mathsf{RSS}_a/(n - n_a n_b)$	
total	$n - 1$		

Read the table backward (starting with the interaction).

- If there is a significant interaction, the main effects are **not** of interest and potentially misleading.

Intuition behind degrees of freedom

The model always includes an overall average μ. There are

- $n_a - 1$ free row means since $n_a\mu = \mu_{1.} + \cdots + \mu_{n_a.}$
- $n_b - 1$ free column means as $n_b\mu = \mu_{.1} + \cdots + \mu_{.n_b}$
- $n_a n_b - (n_a - 1) - (n_b - 1) - 1$ interaction terms

B **ending** A **complexity**	b_1 **(happy)**	b_2 **(sad)**	***row mean***
a_1 (complicated)	μ_{11}	X	$\mu_{1.}$
a_2 (average)	μ_{21}	X	$\mu_{2.}$
a_3 (easy)	X	X	X
column mean	$\mu_{.1}$	X	μ

Terms with X are fully determined by row/column/total averages

Example 1

The interaction plot suggested that the two-way interaction wasn't significant. The F test confirms this.

There is a significant main effect of both `purchase` and `debttype`.

term	**SS**	**df**	**F stat**	**p-value**
purchase	752.3	1	98.21	< .001
debttype	92.2	1	12.04	< .001
purchase:debttype	13.7	1	1.79	.182
Residuals	11467.4	1497		

Example 2

There is a significant interaction between `station` and `direction`, so follow-up by looking at simple effects or contrasts.

The tests for the main effects are not of interest! Disregard other entries of the ANOVA table

term	**SS**	**df**	**F stat**	**p-value**
station	75.2	3	23.35	< .001
direction	0.4	1	0.38	.541
station:direction	52.4	3	16.28	< .001
Residuals	208.2	194		

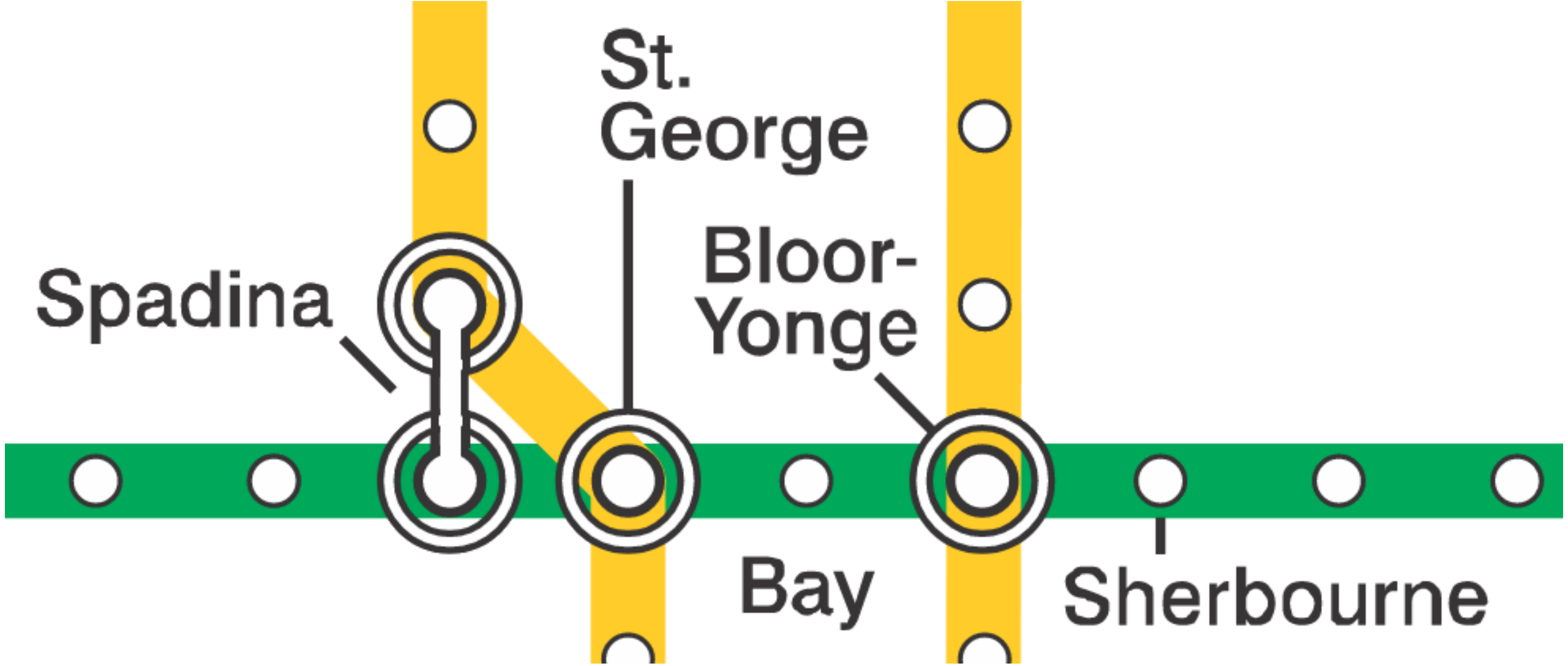
Main effects for Example 1

We consider differences between debt type labels.

Participants are more likely to consider the offer if it is branded as `credit` than loan. The difference is roughly 0.5 (on a Likert scale from 1 to 9).

```
## $emmeans
##  debttype emmean    SE   df lower.CL upper.CL
##  credit     5.12 0.101 1497     4.93     5.32
##  loan       4.63 0.101 1497     4.43     4.83
##
## Results are averaged over the levels of: purchase
## Confidence level used: 0.95
##
## $contrasts
##  contrast      estimate    SE   df t.ratio p.value
##  credit - loan    0.496 0.143 1497   3.469  0.0005
##
## Results are averaged over the levels of: purchase
```

Toronto subway station



Simplified depiction of the Toronto metro stations used in the experiment, based on work by Craftwerker on Wikipedia, distributed under CC-BY-SA 4.0.

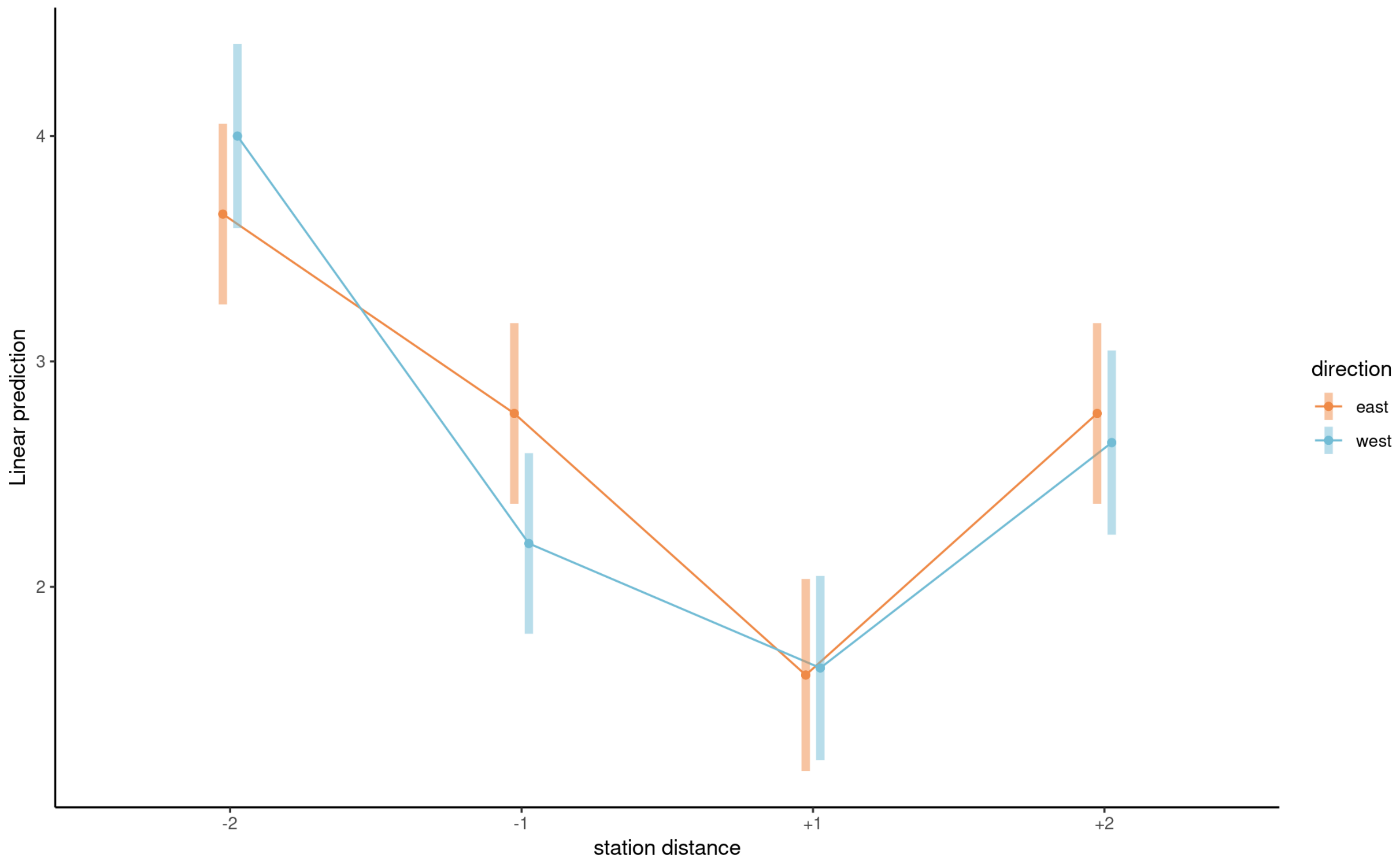
Reparametrization for Example 2

Set `stdist` as $-2, -1, +1, +2$ to indicate station distance, with negative signs indicating stations in opposite direction of travel

The ANOVA table for the reparametrized models shows no evidence against the null of symmetry (interaction).

term	SS	df	F stat	p-value
stdist	121.9	3	37.86	< .001
direction	0.4	1	0.35	.554
stdist:direction	5.7	3	1.77	.154
Residuals	208.2	194		

Interaction plot for reformated data



Custom contrasts for Example 2

We are interested in testing the perception of distance, by looking at $\mathscr{H}_0 : \mu_{-1} = \mu_{+1}, \mu_{-2} = \mu_{+2}$.

```
mod3 <- lm(distance ~ stdist * direction, data = MP14_S1)
(emm <- emmeans(mod3, specs = "stdist"))
  # order is -2, -1, 1, 2
contrasts <- emm |> contrast(
  list("two dist" = c(-1, 0, 0, 1),
       "one dist" = c(0, -1, 1, 0)))
contrasts # print pairwise contrasts
test(contrasts, joint = TRUE)
```

Estimated marginal means and contrasts

Strong evidence of differences in perceived distance depending on orientation.

```
##  stdist emmean    SE  df lower.CL upper.CL
##  -2       3.83 0.145 194     3.54     4.11
##  -1       2.48 0.144 194     2.20     2.76
##  +1       1.62 0.150 194     1.33     1.92
##  +2       2.70 0.145 194     2.42     2.99
##
## Results are averaged over the levels of: direction
## Confidence level used: 0.95
```

```
##  contrast estimate    SE  df t.ratio p.value
##  two dist   -1.122 0.205 194  -5.470  <.0001
##  one dist   -0.856 0.207 194  -4.129  0.0001
##
## Results are averaged over the levels of: direction
```

```
##  df1 df2 F.ratio p.value
##    2 194  23.485  <.0001
```